Theory of Orthogonal Polynomials and Applications

Application for CIM PhD course 2026 HT

1. Course information

Course period: **Period 2**, HT 2026
Extent (credits): 5hp
Contact person: Rostyslav Kozhan, kozhan@math.uu.se
Course structure: 15 lectures (90min each)
Assessment: Hand-in assignments and/or lecture presentation
Location: Uppsala campus
Language of instruction: English

2. Course description

Orthogonal polynomials are essential tools across applied mathematics, computational science, and physics. This course provides a rigorous foundation in the modern theory of orthogonal polynomials, moving beyond classical families to explore general orthogonality on the real line and on the complex unit circle. The theoretical core focuses on the interplay between recurrence coefficients (Jacobi and Verblunsky) and the spectral properties of the associated Hermitian and unitary matrices.

By developing analytical tools such as the Christoffel–Darboux kernel, Herglotz, Carathéodory, Schur functions, and asymptotic theory (Szegő's and Rakhmanov's theorems), the course builds a bridge to concrete applications. Students will learn how these mathematical structures govern the behaviour of numerical methods (Padé approximation, Gaussian quadrature, Lanczos and qd-algorithms), physical interpretations (electrostatics, quantum mechanics) and models (discrete Schrödinger equations, open quantum systems), and information science (signal processing in autoregressive time series). The objective is to provide doctoral students with a shared, robust mathematical framework applicable to their own research.

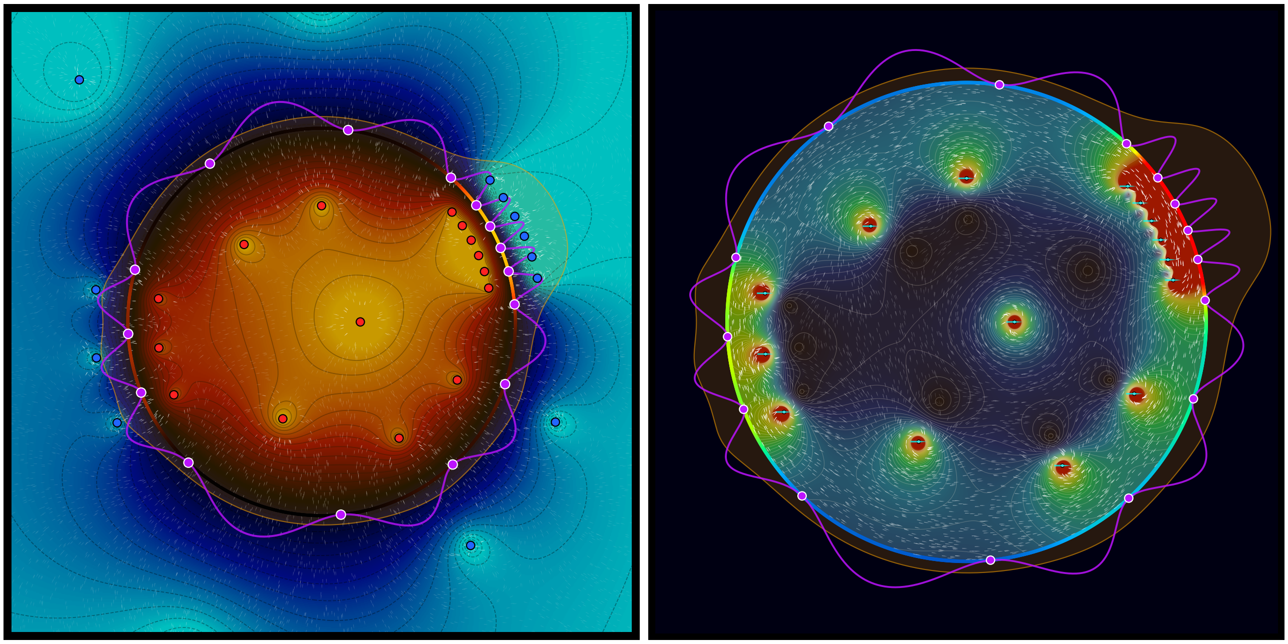
3. Course content

Theory:

- **Foundations**: Orthogonality on the real line and the unit circle; Jacobi and Verblunsky recurrence coefficients.
- **Operator and Spectral Theory**: Jacobi and CMV matrices (including discrete Schrödinger operators); Spectral Theorem for Hermitian and unitary matrices.
- **Analytic Properties**: Properties of zeros, second-kind polynomials, and the Christoffel–Darboux kernel.
- **Functional Analysis**: Herglotz, Carathéodory, and Schur functions; Toeplitz and Hankel determinants.
- **Classical Families**: Hermite, Laguerre, Jacobi, Chebyshev of the first and second kinds.
- **Asymptotics**: the Nevai class; Rakhmanov's and Weyl's theorems, ratio and root asymptotics; Szegő's theorems.
- **Modern Perspectives**: multiple orthogonality, matrix-valued orthogonal polynomials, quasi-definite systems, Baxter polynomials.

Applications:

- **Numerical Analysis**: Padé approximation, Gaussian quadrature, and the Lanczos / qd-algorithms.
- **Physics**: Electrostatic and fluid dynamics interpretations, discrete Schrödinger equations, the harmonic oscillator, and quantum systems via Random Matrix Theory.
- **Information Science**: Signal processing and prediction in autoregressive (AR) time series.



(A) An electrostatic representation of zeros of paraorthogonal polynomials. Zeros of orthogonal polynomials act as fixed point charges.

(B) A dual fluid dynamics interpretation of zeros of paraorthogonal polynomials. Zeros of orthogonal polynomials act as vortexes.

FIGURE 1. Visual representations of orthogonal polynomial zeros.

4. Learning outcomes

The aim of this course is to provide a rigorous mathematical foundation in the theory of orthogonal polynomials on the real line and the unit circle, bridging abstract operator theory with concrete applications. Upon completion of the course, the student should be able to:

- Formulate the definitions and fundamental properties of general orthogonal polynomials on the real line and the unit circle.
- Derive and apply the three-term and Szegő recurrence relations, and critically analyze the properties of their associated recurrence coefficients and zeros.
- Connect the algebraic properties of orthogonal polynomials to the spectral theory of Jacobi and CMV matrices, and utilize this framework to analyze corresponding spectral measures.
- Apply advanced analytical tools, such as the Christoffel–Darboux kernel and Herglotz/Carathéodory/Schur functions, to evaluate the properties of orthogonal polynomials.
- Characterize the classical families (Hermite, Laguerre, Jacobi, Chebyshev) through their orthogonality measures, differential equations, Rodrigues' formulas, and structure relations.
- Relate the theory of orthogonal polynomials to the evaluation of Toeplitz and Hankel determinants, and apply Szegő's theorem in its various formulations.
- Identify how the theoretical framework of orthogonal polynomials is applied to solve problems in numerical analysis, physical sciences, and signal processing.

5. Preliminary Course Structure

The course will be implemented in Period 2, Fall 2026. There will be ~ 10 lectures with theory, and ~ 5 lectures with various applications.

First lecture is planned for Nov 5, 2026, 13:15-15:00.

Dates: Lectures will be held in room 64119 unless otherwise indicated. Tentative schedule:

L01 Thu Nov 05, 13:15-15:00
L02 Fri Nov 06, 10:15-12:00
L03 Thu Nov 12, 13:15-15:00
L04 Fri Nov 13, 10:15-12:00
L05 Thu Nov 19, 13:15-15:00
L06 Fri Nov 20, 10:15-12:00
L07 Mon Nov 23, 10:15-12:00
L08 Thu Nov 26, 13:15-15:00
L09 Fri Nov 27, 10:15-12:00
L10 Mon Nov 30, 10:15-12:00

L11 Thu Dec 03, 13:15-15:00
L12 Fri Dec 04, 10:15-12:00
L13 Mon Dec 07, 10:15-12:00
L14 Thu Dec 10, 13:15-15:00
L15 Fri Dec 11, 10:15-12:00

6. Evaluation

To pass the course students will be offered two options:

(i) Submit written solutions to (theoretical) problems.
(ii) Choose one of the topic with application of orthogonal polynomials (in pure math or other sciences), self-study, and prepare a 30min presentation for one of our December lectures. A list of potential topics will be offered. Course participants may also propose articles outside the list.

7. Registration

To register for the course and for additional information please contact Rostyslav Kozhan kozhan@math.uu.se.

PhD/master students from all departments are welcome (for bachelor/master students this would be counted a selected topics course).

8. Recommended Literature List

The course material will be drawn from selected chapters from the standard texts listed below, as well as the complementary recent research papers.

- B. Simon: Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials (Princeton University Press, 2011)
- Mourad H. Ismail: Classical and Quantum Orthogonal Polynomials in One Variable (Cambridge University Press, 2005)
- B. Simon: Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory (AMS 2005)
- B. Simon: Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory (AMS 2005)

9. Recommended Prerequisites

Familiarity with linear algebra, elements of complex analysis and probability theory.